

A mechanistic model of separation bubble

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Abstract

This work uncovers the low-dimensional nature the complex dynamics of actuated separated flows. Namely, motivated by the problem of model-based predictive control of separated flows, we identify the requirements on a model-based observer and the key variables and propose a prototype model in the case of thick airfoils as motivated by practical applications.

The approach in this paper differs fundamentally from the logic behind known models, which are either linear or based on POD-truncations and are unable to reflect even the crucial bifurcation and hysteresis inherent in separation phenomena. This new look at the problem naturally leads to several important implications, such as, firstly, uncovering the physical mechanisms for hysteresis, secondly, predicting a finite amplitude instability of the bubble, and thirdly to new issues to be studied theoretically and tested experimentally. More importantly, by employing systematic reasoning, the low-dimensional nature of these complex phenomena at the coarse level is revealed.

Key words: separation bubble, separation control, low-dimensional modeling, phenomenology, catastrophe theory

1 Introduction and methodology

1.1 History and motivation

It is known that dynamic vortex shedding can lead to losses in lift, sharp increases in drag, and destructive pitching moments and buffeting, which all limit an aircraft flight envelope. Therefore, in order to improve aerodynamic characteristics, flow separation control would be highly desirable. The classical approach—an open-loop control achieved either by mechanical or fluidic

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actuation—has demonstrated robustness, but its efficiency is still far from optimal. This standard control scheme is based on actuator operating schedules, which are usually constructed using extensive and costly experimental studies.

Alternatively, from a theoretical point of view, should one be able to construct an accurate solution of the Navier-Stokes description (NSEs) for a given airfoil shape and flow conditions, it would suggest control strategies. However, in view of the impossibility of solving the NSEs in real time and in view of noisy and unpredictable real conditions, this approach is difficult to implement. At the same time, in reality one can use sensors on the boundary of lifting surfaces, which in turn read off a certain amount of extra information from the physical system and therefore should allow one to weaken the requirements on the accuracy of theoretical prediction of the flow behavior. Thus, one is naturally led to consider *coarse* models.

However, it should be kept in mind that the dynamical behavior of the original and coarsened (reduced) system will never be identical, and thus one has to decide, based on the application objectives, which aspects of the dynamics should be modelled accurately. In this work we identify the crucial elements of the dynamics of separation bubble, namely *bifurcation* and *hysteresis*, which need to be reflected in the model and thus result in the applicability of the model to a wide range of physical parameters. This procedure is targeted to produce a model, upon which an observer in a closed-loop control scheme can be based. Being more efficient and reliable [1], feedback control also naturally allows one to address the optimization issue.

While the above is a transparent justification to appeal to coarse modeling, the main challenge is that the resulting model should be both *low-dimensional*, for real time computational efficiency, and *physically motivated*, in order to reflect the actual behavior for a wide range of flight and control parameters. Since separation phenomena are clearly nonlinear, the model should also be *nonlinear*. These are certainly *key requirements* on a model.

While these *key requirements* are readily appreciated, the methods available to formulate such models are very limited and the connection of known models to physics is rather far from what is desired. A commonly used approach is to first generate experimental data and, then, to extract the model by a projection onto proper orthogonal decomposition (POD) modes using, for example, balanced truncation or similar methods, is not reliable in view of the open flow nature of the problem and the wide range of governing parameters.

Because of this, we shall make use of *phenomenological* modeling, which has been successful in many other problems, such as the use of Duffing’s equation for the buckling of elastic beams [2], simple maps to describe a dripping faucet [3], which even captures the observed chaotic behavior to a great extent, and

bubble dynamics in time periodic straining flows [4], to name a few. The phenomenological approach was already used in the construction of the first few models for separation phenomena, *e.g.* [5] and the ONERA model [6], after the recent understanding of the importance of low-dimensional models for controlling separation. The state-of-the-art low dimensional model used in a closed-loop control of the dynamic stall with pulsed vortex generator jets is due to Magill *et al.* [5]. Its key feature is a choice of the governing physical parameters, such as lift Z and separation state B with $B = 0$ corresponding to fully attached flow and $B = 1$ to fully separated flow. Steady states, $B_s(\alpha)$ and $Z_s(\alpha)$, represent the baseline case and the measured steady lift, respectively, as functions of the angle of attack α . The experimentally measured function $Z_s(\alpha)$ which may contain a hysteretic behavior and thus is an empirical way of accounting for a hysteresis, as suggested by Magill *et al.* [5]. Exploiting the physical arguments: (i) lift $Z \sim$ circulation $\Gamma(\alpha)$; (ii) relaxation to a baseline state $\lim_{t \rightarrow +\infty} B(t) = B_s(\alpha)$; (iii) rise in lift when a dynamic vortex is shed $Z \sim B_t$, one arrives at the simplest low-order model with adjustable parameters,

$$B_{tt} = -k_1 B_t + k_2 [B_s(\alpha) - B], \quad (1a)$$

$$Z_t = k_3 B_{tt} + k_4 [Z_s(\alpha) - Z] + \Gamma_\alpha \alpha_t. \quad (1b)$$

It should be stressed that this and many earlier attempts to develop dynamical models are based on the ansatz that this nonlinear phenomenon behaves linearly for small variations of the parameters involved [5, 6, 7], which clearly has many limitations, in particular cannot account for bifurcations and hysteresis. Thus, only with an alternative approach—the subject of this work—can one construct a model that meets the above *requirements*. As it will be clear from the text later, while we appeal to phenomenological analysis of empirical facts, we provide the dynamical systems grounds for it. A symbiosis of these two methodologies yield a complete picture of the phenomena.

1.2 Central idea, methodology, and paper outline

A central notion and object, whose dynamics we study, is a *separation bubble*, whose main features are as follows. First of all, separation of the boundary layer develops due to an adverse pressure gradient [8] which occurs when the angle of attack of an airfoil is sufficiently large, cf. Figure 1(a), and may be followed by re-attachment as in Figure 1(b), thus forming a typical flow around an airfoil. The region encompassed by the boundary layer is termed a *separation bubble* after the work of Jones [9] and, as shown in Figure 1, it can be closed or open. Classification of separation bubbles concerns their laminar or turbulent nature, but topologically they do not differ and thus we will not be distinguishing between various cases, but rather treat a *generic* case. It should be noted that, in certain physical situations, a bubble needs to

be understood in a time-averaged sense [10]. Given the notion of a separation bubble, our dynamical systems model will aim to capture its characteristics, which are important for controlling separation phenomena.

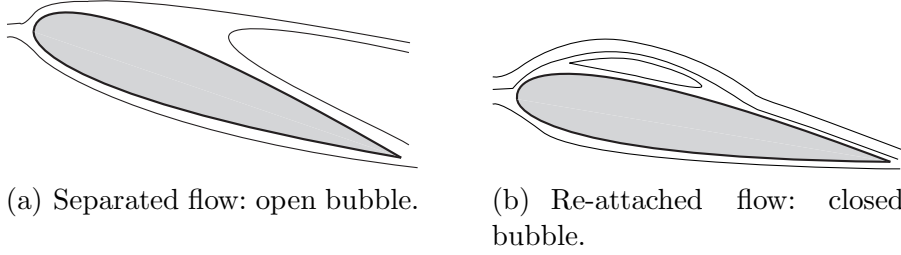


Fig. 1. On the notion of separation bubble.

A central idea of this work is to approach the modeling of separation bubble phenomena by identifying the key crucial elements of the bubble dynamics, namely bifurcations and hysteresis, in the appropriate portion of parameter space, as sketched in Figure 2. In this Figure we show the minimal dimension of the parameter space, defined by the bubble size x , the angle of attack α , and the actuation amplitude w ; that is, we will be looking for the minimal model determined by the dependence of the bubble size on the angle of attack and actuation amplitude.

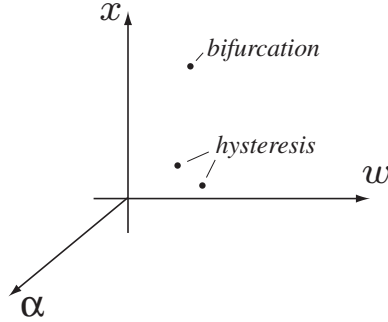


Fig. 2. A cartoon of the key dynamic elements—bifurcation and hysteresis—to be captured by the minimal number of parameters, namely the bubble size x , the angle of attack α , and the actuation amplitude w .

This minimal approach is motivated by the fact that while generally there are other parameters involved, such as the Reynolds number Re , the critical angle of attack α_c , and the airfoil thickness h , the resulting model will still have wide applicability. This can be understood based on the aerodynamic properties of airfoils. To explain this, we draw critical curves, i.e., when separation takes place depending upon Re , α_c , and h in Figure 3.

As illustrated by Figure 3(a), in the case of real airfoils, separation occurs at finite Reynolds numbers even at zero critical angle of attack; the higher α_c the lower the critical Reynolds number Re_c ; also, the thicker the airfoil,

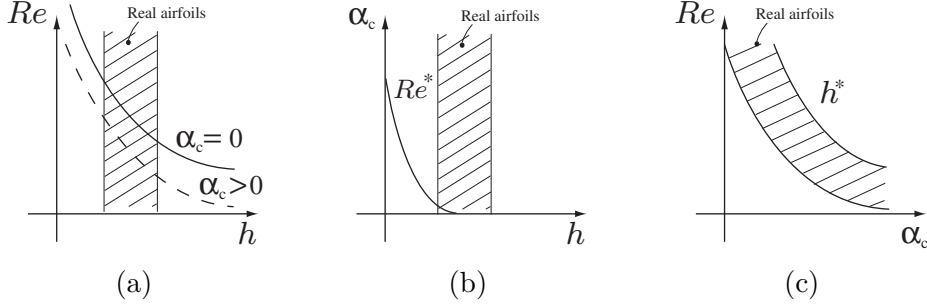


Fig. 3. The placement of real airfoils in the parameter space defined by the Reynolds number Re , the critical angle of attack α_c , and the airfoil thickness h : the critical curves corresponding to the instant when separation occurs. Re^* and h^* are typical fixed values of these parameters.

the lower Re_c . Figure 3(b) demonstrates the fact that the thinner an airfoil, the larger the critical angle of attack is required to achieve separation at a given Reynolds number Re^* . Finally, in the α_c - Re plane in Figure 3(c) one can observe that for fixed airfoil thickness h^* separation can occur at zero α_c , which requires high enough Reynolds numbers. Since in reality the Reynolds numbers are huge (e.g. for real aircraft Re varies between 10^6 and 10^{11}), one concludes that limiting ourselves to “thick airfoils”, which can, in fact, be regarded as real airfoils since they have to carry structural load and fuel, is not a serious restriction in this first step towards low-dimensional modeling of separation phenomena.

To achieve the above objectives of our modeling identified above, we will appeal to the tools of the bifurcation and catastrophe theory [11], as will be made precise in §2. The outline of the paper is as follows. In §2, we discuss the first nonlinear aspect of separation bubbles, namely bifurcation phenomenon and the way to model it. In §3, we explore the basic physics of hysteresis phenomena, and suggest its mathematical model and how to construct a single model capable of capturing both bifurcation and hysteresis.

2 Bifurcation in the dynamics of separation bubble

2.1 On the notion of bifurcation

As was noted in the introduction, bubbles can be either in a closed or open state. This allows us to introduce a key element of the low-dimensional modeling, namely it must capture this *basic bifurcation* from an *open* to a *closed* state, as shown schematically in Figure 4, which is also known as *bursting* [12].

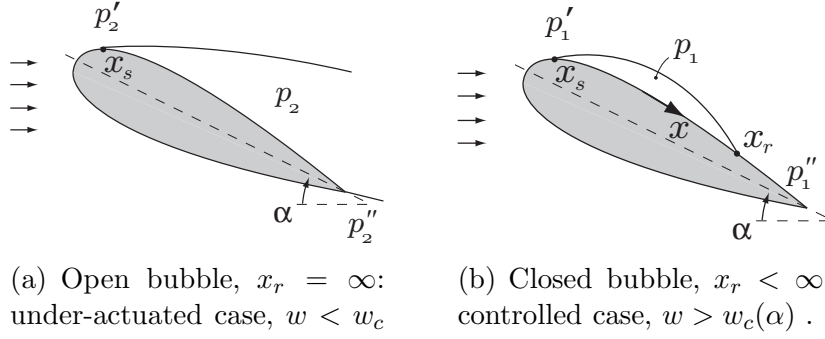


Fig. 4. Basic setup and primary bifurcation.

Notably, the fact that this is the primary bifurcation was realized just recently [13]. The vast literature on separation bubbles behavior is still at a descriptive level and suggests that one separated flow is not like any other. Here we take a different point of view, i.e. we treat the coarse behavior of separation bubbles as (generic) phenomena that can be modelled by a single low dimensional dynamical system.

2.2 Quantifying separation bubbles

To quantify the behavior of a separation bubble, consider the coordinate x , measuring the distance along the airfoil from the bubble onset to the bubble reattachment, as shown in Figure 4. The bubble dynamics in the first approximation can be described by two parameters: the location of separation, x_s , and of reattachment, x_r , which can move under the change of flight and control parameters. In some cases, e.g. the Glauert Glas II airfoils, the separation point x_s remains fixed for all practical purposes. Therefore, we will start by considering only the behavior of the reattachment point, which experiences a primary bifurcation in the above sense; extending the model to include variation of x_s will require the addition of a reliable separation criterion. As an alternative to x_r , one could also utilize the bubble area. From now on we will use x as a variable representing the bubble state.

2.3 On the physical nature of bifurcation

The mechanism by which the excitation affects the flow lies in the generation of instabilities, and thus of Large Coherent Structures transferring high momentum fluid towards the surface, and therefore leading to reattachment, as indicated in Figure 5. Since actuation exploits the instabilities of the shear layer [14], the response to actuation depends on both w and ω and therefore is nonlinear. The latter again indicates, now from the point of view of actuation

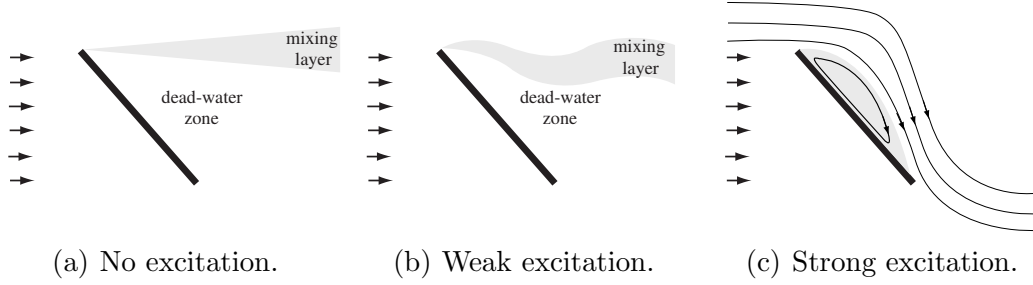


Fig. 5. On the mechanism of actuation.

control mechanisms, that the low-dimensional model must be nonlinear.

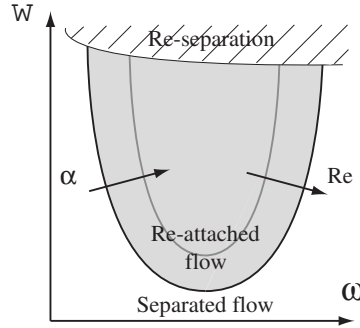


Fig. 6. Effect of time-variant actuation: criticality of actuation amplitude w and frequency ω . Shaded region corresponds to reattached flow (closed bubble). Arrows indicate the change in location of the transition curve with increasing Re and α .

As follows from experiments, the critical phenomena are as sketched in Figure 6, where the shaded region corresponds to a reattached flow (that is, a closed bubble) and the arrow indicates a change in location of the transition curve with an increase in α . The size of the bubble, x , has a specific dependence on the amplitude w and frequency ω of actuation, *i.e.* $\partial x / \partial w < 0$, $\partial x / \partial \omega < 0$, when moving away from the origin $(w, \omega) = \mathbf{0}$ in Figure 6. In this work we focus on the case of time-invariant actuation, $\omega = 0$, although the time-varying case will be commented on later in this section.

Finally, it is notable that the criticality and hysteresis phenomena depend on the connectedness of the flow domain: the bubble experiences bifurcation only in the case of flow around an airfoil, as in Figure 7(b), while in the case of a hump model in Figure 7(a), which is frequently used in experiments, there is no bifurcation. Thus, there are two basic configurations in which the behavior of the separation bubble differs: the *hump model* and the *airfoil model*. Namely, in the hump case $x(w)$ is smooth, while in the case of an airfoil $x(w)$ is discontinuous. Also, as will be important in §3, the hysteresis phenomena are present only in the airfoil case. Here, in view of its practical importance, we naturally focus on the airfoil case.

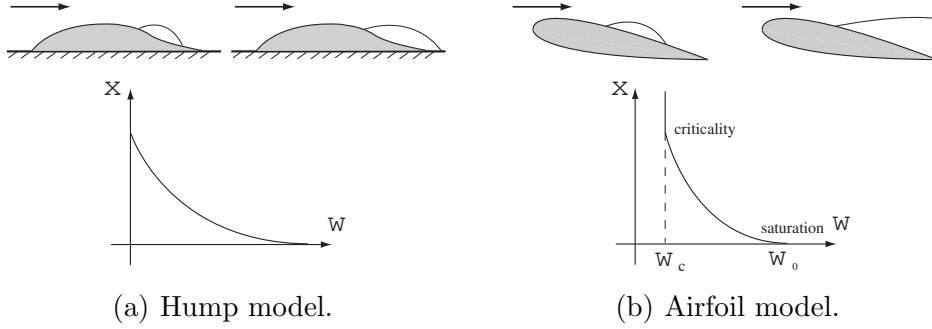


Fig. 7. Two basic experimental configurations.

2.4 Modeling the bubble bifurcation

In developing a model, we are guided by the principle of a minimal complexity together with the physical requirements one has to meet. At the methodological level, there are two basic ways to account for the form of $x(w)$, which has both the saturation and criticality shown in Figure 7: (a) to design $F(x, w) = 0$ as an *algebraic* relation, or (b) to introduce a *dynamic* description $F(x, \dot{x}, \ddot{x}, \dots, w)$. The latter approach is better suited for dynamics and control purposes, because in the case of active feedback control one would need to deal with a few characteristic times and transient effects, and thus the model should be time-dependent. The simplest possible way of introducing time-dependent dynamics is a second-order oscillator model, $\ddot{x} - \mu\dot{x} = F(x, w)$, where μ is a damping parameter. The justification for the latter may serve the fact that both separation and reattachment points may oscillate [10].

In what follows, we first formulate mathematical requirements on a model in §2.4.1, then by appealing to the ideas of a potential function in §2.4.2 and a dynamic bifurcation in §2.4.3, we construct the model in §2.4.4.

2.4.1 Mathematical requirements

Naturally, the bubble size x also depends on a *flight parameter*, in our case the angle of attack α , which needs to be incorporated in the model; thus, $F = F(x, w, \alpha)$. Since we want to minimize the functional complexity, but to retain the nonlinear features of the phenomena, the simplest form is a quadratic nonlinearity, $F(x, w, \alpha) = x^2 + b(w, \alpha)x + c(w, \alpha)$, which possesses a Takens-Bogdanov bifurcation, as shown in Figure 8, when $b^2 - 4c$ changes sign.

Indeed, equilibria points are given by $x_{1,2} = -\frac{b}{2} \pm \frac{1}{2}\sqrt{b^2 - 4c}$, so that F can be represented as $(x - x_1)(x - x_2)$. The eigenvalues of the linearization around $x = x_1$ are given by $\lambda_{1,2}^2 = x_1 - x_2 = \sqrt{b^2 - 4c}$, while the eigenvalues of the linearization around $x = x_2$ are $\lambda_{1,2}^2 = x_2 - x_1 = -\sqrt{b^2 - 4c}$. Thus when $b^2 - 4c$

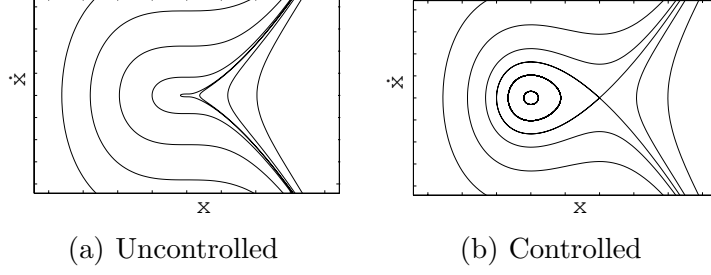


Fig. 8. Takens-Bogdanov bifurcation.

changes sign, one observes the transition from the picture in Figure 8(a) to the one in Figure 8(b). The requirements on the parameters in this model are dictated by the physics:

(a) The stability of equilibria points should obey

$$\begin{aligned}
 \alpha < \alpha_c : \quad & b^2 - 4c > 0 \text{ (stability),} \\
 \alpha > \alpha_c : \quad & w > w_c(\alpha), \quad b^2 - 4c > 0 \text{ (stability : no separation),} \\
 & w < w_c(\alpha), \quad b^2 - 4c < 0 \text{ (instability : separation),}
 \end{aligned}$$

where stability implies that one equilibrium is stable (λ is imaginary), and the second one is unstable (λ is real). The above inequalities indicate that the physical behavior of the model is also governed by the critical angle of attack α_c , when the flow separates at $w = 0$, and the critical control amplitude $w_c(\alpha)$, when flow reattaches at α fixed.

(b) The critical actuation amplitude w_c should grow with $(\alpha - \alpha_c)$, since the higher the angle of attack, the more control input is required to make the flow reattached.

(c) The bubble size x , which is a stable equilibrium, should shrink, $x \rightarrow 0$ as w increases. At the same time, the domain of attraction of this equilibrium should shrink too, so that the bubble becomes susceptible to the finite-amplitude instabilities, as it is known from experiments, cf. the upper part of Figure 6.

2.4.2 Potential function approach

In order to get better insight in the model construction, let us assign a potential function $V(x)$ such that $V'(x) = -F(x)$:

$$V(x) = \frac{x^3}{3} + b(w)\frac{x^2}{2} + c(w)x + d(w), \quad (2)$$

which is physically determined by the elastic properties of a bubble and its interaction with the outer flow. Then we can observe that a finite bubble corresponds to $V(x)$ as in Figure 9(a), and an infinite bubble corresponds to Figure 9(b).

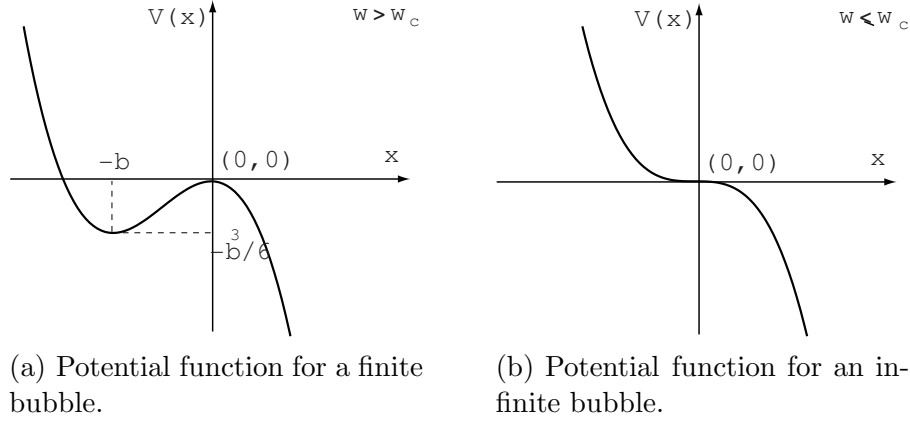


Fig. 9. Potential function $V(x)$; $d = 0$, $c = 0$ in (2).

Without loss of generality, we can assume that $d = 0$. Considering w , as a control parameter, the requirements on the coefficients in $V(x)$ are such that the equilibria, $V'(x) = -x^2 - bx - c = 0$, obey

- $w > w_c$: two equilibria (stable and unstable), $V''(x_1) > 0$, $V''(x_2) < 0$;
- $w \leq w_c$: only one equilibrium point, which is unstable (marginally stable).

The stability conditions can also be reformulated in terms of eigenvalues, as indicated in Figure 10. In this particular case, the equilibria points $x_i(w, w_c)$ are easily computable: $x_1 = -b$ and $x_2 = 0$. The stability criterion (second variation) for these equilibria is given by the sign of the second derivative, $V''(x) = -2x - b$, which at the equilibria points assumes the values $-b$ and b , respectively. Besides the stability conditions, one needs to impose $dx_1/dw < 0$, since the bubble shrinks when the control amplitude increases. Thus, the bifurcation from the state in Figure 9(a) to the one in Figure 9(b) is obviously associated with the condition when $b(w_c) = 0$. As one can further infer, in the space of curves in (w, w_c) there is an infinite number of solutions $b = b(w, w_c)$, $c = c(w, w_c)$. In practice, a *systematic procedure* would be as follows: depending on the particularities of the experimental data, one expands b and c in terms of some basis functions of w , w_c , etc., and then determines coefficients in that expansion through the calibration procedure.

2.4.3 Dynamic bifurcation

The transition from one potential to another is controlled by a bifurcation parameter, such as angle of attack α or actuation amplitude w . In fact, the latter two parameters are interchangeable to a certain extent as argued in [15], since a change in α or in w leads to a change in circulation around an airfoil, and thus to a change in the flow structure. Apparently, this transition of x from finite to infinite is *dynamic* in a sense that the bubble becomes infinite in Figure 9(b) as time $t \rightarrow \infty$. This *dynamic bifurcation* can be clarified using phase por-

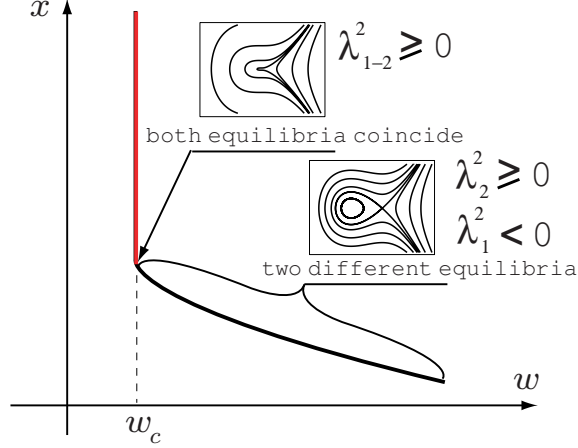


Fig. 10. Critical curve in the (x, w) -plane: on the dynamic bifurcation; solid black line represents stable equilibria, solid red line is a dynamic bifurcation when bubble grows indefinitely with time. Phase portraits in rectangles correspond to the ones in Figure 8.

traits in Figure 10, and should be opposed to the standard static bifurcation, which is of algebraic nature as resulting from the condition of vanishing vector fields. As one can learn from Figure 10, at the critical value of the actuation amplitude w_c both equilibria coincide and are unstable (marginally stable), so that the bubble grows with time and becomes unbounded for $t \rightarrow \infty$. For $w > w_c$ there are two equilibria points, one is unstable and one is stable. The latter corresponds to the situation when bubble is of finite size, and this state has a finite domain of attraction. Note that the potential energy shape, as in Figure 9, is crucial in allowing the “dynamic” bifurcation: a V -shaped potential function apparently would not allow this type of bifurcation, as well as the domain of attraction would be modelled inconsistently with physics. Similar type of argument will be applied to the hysteresis phenomena in §3. In conclusion, having identified, based on the physical argumentation, that the potential should be of the shape as in Figure 9 in order to allow a dynamic bifurcation, the problem reduces to determination of the coefficients in (2). This general procedure is the subject of the catastrophe theory [11] and, at the technical level, is in the realm of calculus [16].

2.4.4 Model and its interpretation

For simplicity, restricting ourselves to the case of thick airfoils when separation occurs at $\alpha_c = 0$ without actuation, with one of infinitely many admissible choices of b and c we get:

$$\ddot{x} = -V_x(x; \alpha, w) = -\mu\dot{x} + (x - \alpha)^2 + f(w)x. \quad (3)$$

Here $f(w) = a_1 w + a_2 w^2 + \dots$ represents a *nonlinear response* of the separation region to actuator excitations w . The product $f(w)x$ implies that the effect of actuation depends upon the bubble size x . As required, $f(w_c) = 2\alpha^{1/2}$ and the bubble shrinks as $\sim f^{-1}$ for $w \rightarrow \infty$. While this is the simplest possibility, from the above description it is clear that there is enough flexibility to calibrate the model through the fitting functions, $F(x, w, \alpha)$, and parameters, (b, c, \dots) within the given above bounds.

By construction, the model (3) reflects the basic generic dynamic behavior of separation bubbles. In the *conservative* time-invariant case the parameter space is just $(\alpha, f(w))$. When control is absent, $f(w) = 0$, the bubble is open, which corresponds to an unstable phase portrait in Figure 8(a), that is any initial conditions lead to an unbounded bubble size x . When sufficient control is applied (consider first α fixed), the bubble closes, which is reflected in the change of the phase portrait as shown in Figure 8(b). In this case there are two equilibrium points, one is a saddle, which is unstable and thus not physically observable, and another one is a stable center. Therefore, there exists a non-zero basin of attraction which leads to a finite bubble size, $x < \infty$. Figure 8(b) also suggests that the system is susceptible to finite-amplitude instability for $w > w_c$, the fact which is conceivable but has never been studied in experiments systematically. Nevertheless, it is known empirically that the bubble opens if the actuation amplitude becomes large enough, as in Figure 6; see also [17]. Also, the fact that the boundary layer is susceptible to finite-amplitude instabilities [18] suggests that the separation bubble formed out of it may also be finite-amplitude unstable. The inclusion of dissipation in the model (3) does not change the nature of the phase portrait; however, it does change the basin of attraction.

Finally, the inclusion of time-varying effects in the control, $w = w_0 \cos \omega t$ with $\omega \neq 0$, also demonstrates that the bubble transforms from an open to a closed state. Thus, as required, the model (3) captures the primary bifurcation and dynamic behavior of the separation bubble, except for the hysteresis. In the rest of this paper we will explain how the model (3) can be *enhanced* to account for the hysteresis shown in Figure 12. While the model (3) is given for one of infinitely many choices of parameters, it is clear that a variety of other admissible choices can produce the same type of bifurcation and dynamics. This freedom to choose parameters is important, however, in order to fit the model to a particular application via calibration.

2.5 Analogy to other physical phenomena

It is notable that a model of a similar form was deduced *ad hoc* for a real bubble deforming in a straining flow studied by Kang & Leal [4], shown in

Figure 11(a), which experiences a bifurcation from a deformed but closed state to an open tip-streaming state, when bubble forms pointed open ends emitting tiny bubbles.

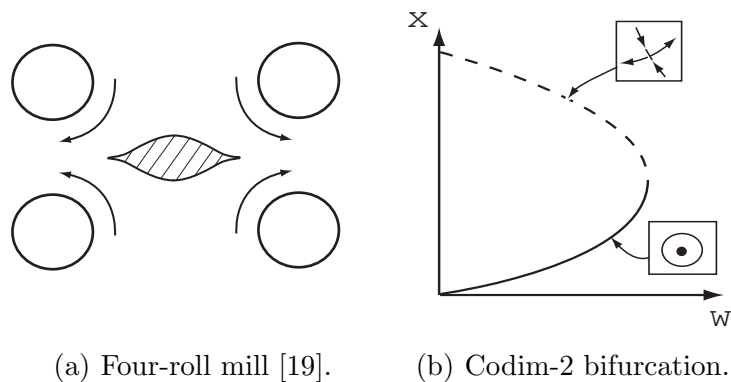


Fig. 11. Bubble deformation in a four-roll mill [4].

Namely, the model is $\ddot{x} = -\mu\dot{x} + (ax - bx^2) + w$, where w is the control parameter (Weber number). The dynamics of this problem is illustrated in Figure 11(b) for the conservative case, $\mu = 0$, and reflects the fact that for the same w there are two equilibria, one of which is a stable center and another one is a saddle; the latter is not observed physically in view of its unstable character. This problem also illustrates the analogy of the dynamics of real and separation bubbles.

3 Hysteresis in the dynamics of separation bubble

3.1 Experimental observations: the model objectives

The basic effects of time-varying control were discussed in §2.3 and reflected in Figure 6. However, the effect of changing amplitude and frequency is not trivial in view of the presence of a hysteresis [20, 21] in all variables (α, w, ω) , as illustrated in Figure 12 for the dependence of the bubble size on the actuation amplitude, $x(w)$. Experiments demonstrate that hysteresis is present no matter how slowly the actuation amplitude w is changed, which suggests that the model should depend only on the sign of the rate \dot{w} .

Therefore, the model should reflect the fact that there are two stable steady state solutions for the range of the control parameter $w_{c1} < w < w_{c2}$, as in Figure 12, which is an experimental fact. Mathematically, this means that the selection between these two solutions is due to the placement of initial conditions in the corresponding domain of attraction. Also, for $w > w_{c2}$, there

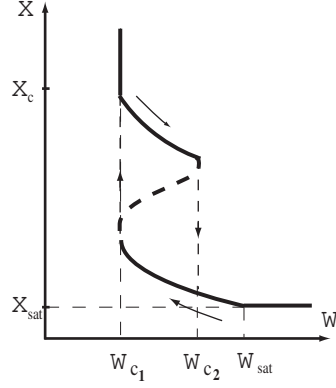


Fig. 12. Experimental effect of time-varying actuation: hysteresis phenomena in amplitude w .

should be only on stable solution, while for $w < w_{c1}$ the bubble should “bifurcate” to infinity in a dynamical manner as described in §2.4. The challenge of modeling the hysteresis comes from the fact that the behavior of the bubble is known only from experimental observations, while there are no analytical results. Clearly, the domain of attraction of stable solutions is not well-characterized from the existing empirical data.

Therefore, in order to model hysteresis, one first needs to understand its physical origin, which is addressed below, in §3.2, where we suggest the physical mechanisms of the hysteresis. This together with the dynamical systems and catastrophe theory allow us to modify the model (3) to account for hysteresis, which is the subject of §3.3.

3.2 On the physics of hysteresis

Physically, the separation bubble is caused by a strong adverse pressure gradient, which makes the boundary layer separate from the curved airfoil surface. Actuation with $w > w_c$ effectively reduces the adverse pressure gradient² and makes the bubble closed, as in Figure 4(b). This can be seen from Bernoulli’s equation, since the velocity drop is related to the pressure rise, $p + \rho u^2/2 = p_0$, where p is a dynamic pressure, and p_0 is the fluid pressure at rest. From Bernoulli’s equation and Figure 4 we can conclude that pressure rise $p'_i - p''_i$ and bubble length l correlate $p'_1 - p''_1 < p'_2 - p''_2$ and $l_1 < l_2$, respectively. For the current purposes we neglect by the second order effects of vorticity and thus assume constant pressure inside the bubble, $p_1 = p_2 = \text{const}$.

² Note that for some airfoils the same effect can be achieved by changing the angle of attack α , i.e., the larger the angle of attack α , the stronger the adverse pressure gradient: this “interchangeability” of the effects of the actuation amplitude and the angle of attack is well-known [15] and is reflected in the dependence $w_c(\alpha)$.

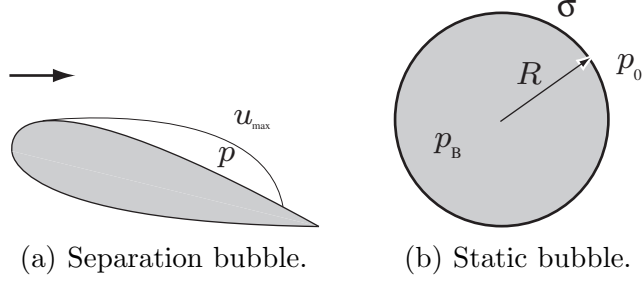


Fig. 13. Contrasting separation and ordinary bubbles.

Let us compare the above behavior of a separation bubble with that of a real static bubble (see Figure 13), which is governed by $p_B = 2\sigma/R + p_0$, where p_B is the pressure inside the bubble, p_0 is the pressure outside the bubble, $\sigma > 0$ is the interfacial tension, and R is the radius of the bubble. Apparently, if the pressure p_0 outside the bubble decreases while the pressure inside is maintained constant, the bubble shrinks³. This behavior of a real bubble, when its pressure inside is maintained constant, contrasts with that of a separation bubble, which grows if the pressure outside the bubble, p , reduces. The underlying physics of these two problems differs: in the first case, the phenomena are dominated by static forces, while separation phenomena are dynamic.

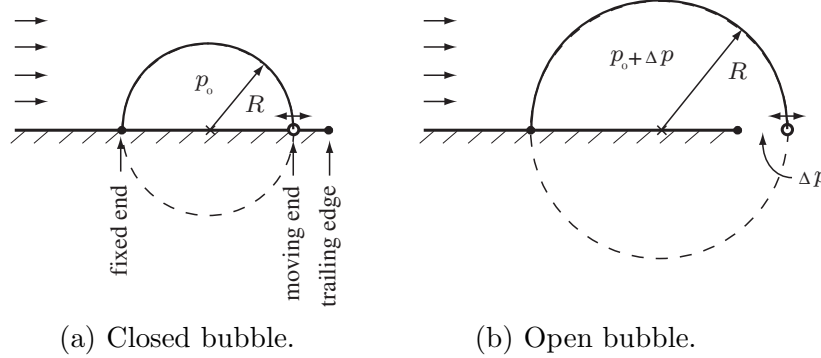


Fig. 14. Mechanical model of separation bubble hysteresis.

This suggests that the separation bubble boundary possesses elastic properties, which for the current purposes can be modelled with negative interfacial tension. Note that real (positive) tension tends to minimize the interfacial area, while the effective tension of the shear layer tends to maximize the bubble boundary and only the external energy input (excitation) counteracts this effect and makes the bubble closed: this justifies a negative sign of the tension. Alternatively, one can use a nontrivial state equation for the pressure inside the bubble, which can be measured experimentally. The elasticity of the separation bubble is evidenced by introducing disturbances outside the

³ In reality, the pressure inside is inversely proportional to the bubble radius which results in bubble growth.

bubble and observing the changes in the bubble characteristics, i.e. shape and pressure inside⁴.

With the above physical background, we can provide a simple mechanistic model explaining the origin of the hysteresis. For simplicity, consider the two-dimensional situation depicted in Figure 14: a hemispherical bubble having variable size with the left end fixed and with its right end free to move thus modeling a separation bubble with moving reattachment point. The bubble size changes depending upon the free-stream velocity u_{\max} , which is chosen to be the control parameter. When u_{\max} increases and the right end of the bubble reaches the trailing edge at R_0 at critical u_{cr}^2 , the pressure inside the bubble increases by a finite amount, $p_0 \rightarrow p_0 + \Delta p_0$, which is due to suction of a high pressure fluid from the lower side of the airfoil. Hence the bubble size increases abruptly by some amount. Conversely, when u_{\max} decreases and bubble reaches the trailing edge at R_0 at a different critical u_{cr}^1 , the pressure inside the bubble relaxes to its original value, $p_0 \rightarrow p_0 - \Delta p_0$. The jump in pressure at the critical point – when reattachment is at the trailing edge – has the following physical explanation. It is known that the lift drops when the bubble opens, which effectively means that the pressure balance between the lower and upper surfaces of an airfoil has changed: some amount of pressure at the lower surface has leaked into the upper surface, namely into the bubble. The latter is allowed by unsteadiness of the process, i.e., the unsteady Kutta-Joukowski condition.

Therefore, the mechanical analog of a separation bubble is $p = p_0 + \tilde{\sigma}/R$, $p > p_0$, so that the bubble grows when the ambient pressure dictated by Bernoulli's equation, $p = p_\infty - \rho u_{\max}^2/2$, decreases:

$$u_{\text{cr}}^2 (\dot{u}_{\max} > 0) : R_0 = \frac{\tilde{\sigma}}{p_\infty - p_0 - \frac{\rho(u_{\text{cr}}^2)^2}{2}}, \quad (4a)$$

$$u_{\text{cr}}^1 (\dot{u}_{\max} < 0) : R_0 = \frac{\tilde{\sigma}}{p_\infty - p_0 - \Delta p_0 - \frac{\rho(u_{\text{cr}}^1)^2}{2}}, \quad (4b)$$

which produces a hysteretic behavior. Obviously, $u_{\text{cr}}^1 < u_{\text{cr}}^2$ is consistent with the physical observations.

In the light of the above, one can account for the hysteresis in Figure 12 in model (3) as follows. When $\dot{w} < 0$ and w passes through w_c the transformation $w \rightarrow w - \Delta w$ with $\Delta w > 0$ is applied, since physically the effectiveness of control drops by Δw . When $\dot{w} > 0$ and $w = w'_c$ then $w \rightarrow w + \Delta w$, since too conservative amount of control has been applied before reaching $w = w'_c$. These altogether lead to the desired hysteretic behavior. Same can be done to account for the hysteretic dependence on the angle of attack α .

⁴ Both elasticity and non-trivial state equation of the separation bubble have been confirmed experimentally (personal communication: John Kiedaisch).

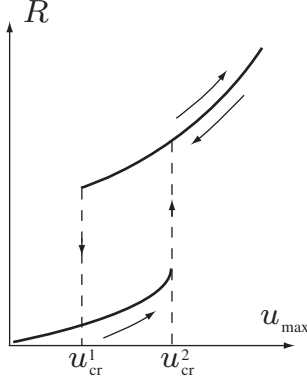


Fig. 15. Schematics of model (4) for hysteresis.

The above mechanistic model of the hysteresis captures the physics and proves that the separation bubble has a nontrivial potential function associated with it. It should be noted that such type of discontinuous modeling of hysteresis based on the rate sign \dot{w} is still widely used in applications and known as *play* and *stop* (classical Prandtl model) models, cf. Visintin [22].

3.3 Accounting for hysteresis in model (3)

However, for the purpose of deriving a universal model which combines both the bifurcation and the hysteresis in a dynamic manner, i.e. suitable for control purposes, it makes sense to follow another way of modeling hysteresis phenomena, based on the choice of an appropriate potential function $V(x)$, similar to what was done in §2.4. We will enforce this point of view in §3.4, where we will illustrate the analogy to other physical phenomena.

The grounding thesis is that the true curve of states in Figure 12 is not the solid discontinuous one, but rather the “true” picture for separation bubbles corresponds to the smooth curve (including the dashed line) in Figure 12, the fact which has not been realized in the literature before. This smooth curve corresponds to the equilibria states of an appropriate potential function $V(x)$; the dashed curve is not physically observable in view of instability of the corresponding equilibria states. From §2.4 we know that the potential function should be of special shape, i.e. when $x \rightarrow \pm\infty$, then $V(x) \rightarrow \mp\infty$, that is the highest order terms in α should be odd. Then, as a natural generalization of the picture in Figure 10, we arrive at Figure 16.

At the technical level, the lowest order potential suitable for achieving the picture in Figure 16 is of the fifth order, so that model (3) becomes

$$\ddot{x}(t) = -V_x(x; \dots), \quad (5)$$

with $V(x)$ of the fifth order. The existence of such potential is apparent, and

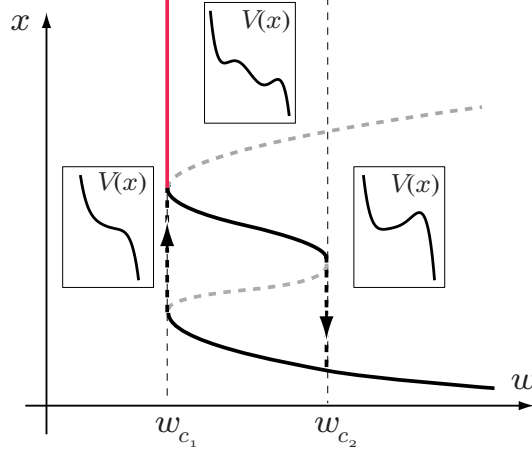


Fig. 16. Hysteresis curve in the (x, w) -plane and corresponding potential functions; solid black lines represent stable equilibria, while dashed lines are unstable equilibria; solid red line represents a dynamic bifurcation (bubble size grows with time unboundedly). Plots $V(x)$ in rectangles show the shape of the potential for $w < w_{c1}$, $w \in [w_{c1}, w_{c2}]$, $w > w_{c2}$, respectively.

its coefficients in the polynomial representation can be found with the help of linear programming given a set of inequalities and equalities based on the calibration requirements.

3.4 Analogy to other physical phenomena

The fact that the hysteresis originates from the particularity of the potential function is well-known from other physical systems, e.g. a ferromagnetic drop deforming in a magnetic field [23] and cavitating hydrofoils [24].

Consider the *deformation of ferrofluid drop* of permeability μ_2 , placed in a fluid of permeability μ_1 , in a magnetic field [23, 25]. The surface energy of the drop is given by

$$E_s = \sigma 2\pi a^2 e \left[e + \epsilon^{-1} \sin^{-1} \epsilon \right], \quad \epsilon = (1 - e^2)^{1/2}, \quad (6)$$

where $e = b/a$ is the aspect ratio, a and b are semi major and semi minor axes respectively, and σ is the interfacial tension. The magnetic energy is of the form

$$E_m = -\frac{V H^2}{8\pi} \frac{\mu_1}{\alpha + n}, \quad \alpha = \frac{\mu_1}{\mu_2 - \mu_1}, \quad (7)$$

where $n = e^2 \{-2\epsilon + \log[(1 + \epsilon)/(1 - \epsilon)]\} / 2\epsilon^3$ is the demagnetization factor, V is the volume of the drop, and H is the applied magnetic field. Minimization

of the total energy, $E_t = E_s + E_m$, with respect to the aspect ratio e produces

$$H^2/\sigma = g(e), \quad (8)$$

the behavior of which is depicted in Figure 17. The bubble shape is a simple counterplay between magnetic and interfacial energy of the drop: the former tends to elongate the drop, while the latter tends to make the drop spherical. For certain values of H^2/σ there are three solutions, but not all of them

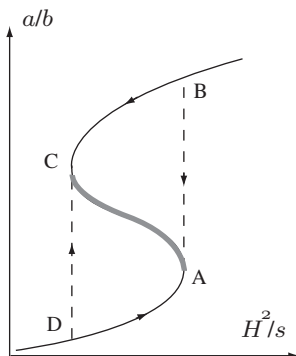


Fig. 17. Deformation of a ferromagnetic drop.

are stable. When the solution reaches point A it jumps to the point B, and similarly for the points C and D. AC portion corresponds to a maximum of E_t and thus is unstable, while the rest of the curve is minima of E_t and thus is stable.

Finally, a *cavitation bubble* on a *hydrofoil*, where hysteresis can be explained with the help of inviscid free-streamline theory [26, 27], is another example, where it has been done analytically. In this physical problem the boundary of the bubble is well-defined physically and thus the problem is reliably treated with a free-streamline theory, that is its predictions [28] agree well with experiments [29]. For the theoretical treatments of cavitation flows with free-streamline theory we refer to Tulin [30], Yeung & Parkinson [31], Birkhoff & Zarantonello [32], and on the physics of cavitation flows to Wu [33], Brennen [34].

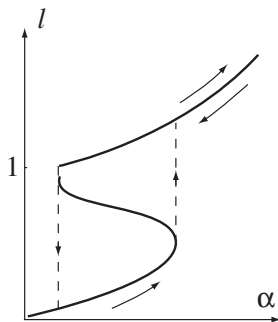


Fig. 18. Cavitating hydrofoil.

On the physical side of cavitation phenomena, it is known from the general equations of fluid dynamics that the pressure depends on the velocity distribution (in the steady case) and on the acceleration (in the unsteady case). More importantly, the pressure might become negative at points where the velocity is large. In the majority of cases, fluids cannot sustain a negative pressure and the continuity of the flow breaks down. As a result, a region filled with fluid vapor is formed—this is a cavitation phenomena (see, for instance, [35]). In continuous incompressible flows the maximum velocity occurs at the boundary⁵ and hence cavitation first appears on the body surface:

$$\chi = \frac{2(p_{st} - p_d)}{2u_\infty^2} = \frac{u_{\max}^2}{u_\infty^2} - 1, \quad (9)$$

which is a cavitation number. Deviations from this law are due to vortex shedding and other unsteady effects. The behavior of the cavitation bubble is given by for partially cavitating, $l < 1$ [26], and supercavitating, $l > 1$ [27], foils respectively,

$$\frac{\chi}{2\alpha} = \frac{2 - l + 2(1 - l)^{1/2}}{l^{1/2}(1 - l)^{1/2}}, \quad l < 1, \quad (10a)$$

$$\alpha \left(\frac{2}{\chi} + 1 \right) = (1 - l)^{1/2}, \quad l > 1, \quad (10b)$$

where α is the angle of attack. The expressions (10) are basically the solution of the equilibrium condition, $V' = 0$, and schematically shown in Figure 18 for fixed cavitation number (see also Sychev [24]).

4 Conclusions

This work has focused on the fundamental aspects—the most important physics and dynamic behavior—of a generic separation bubble using thick airfoils as a paradigm. Given an incomplete experimental knowledge of the complex phenomena of separation bubble, we applied the deduction based on bifurcation and catastrophe theory and thus (1) filled in incomplete pieces in the dynamical picture of the phenomena, (2) advocated that this dynamical picture is finite-dimensional at the coarse level, (3) developed a constructive way of building a model, and (4) produced a model.

The model can be enhanced in particular by (a) incorporating a non-trivial state equation of a bubble, (b) accounting for separation at non-zero angle of

⁵ This follows from the maximum-modulus theorem [36], which states that maxima of a harmonic function must occur on the boundary, but not in the interior of the region.

attack $a_c \neq 0$, and (c) calibrating the model for a given airfoil. These are the future directions of this study and will require considerable theoretical and experimental efforts. We also expect that this approach to low dimensional modeling will be helpful in real time flow control.

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